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Analytical function theory approach to the heat transfer problem of a cylinder in cross-flow at small Péclet numbers

Volfango Bertola [a,*], Emilio Cafaro [b]

[a] University of Edinburgh, School of Engineering and Electronics, The King's Buildings, Mayfield Road, Edinburgh EH9 3JL, United Kingdom
[b] Politecnico di Torino, Dipartimento di Energetica, Corso Duca degli Abruzzi 24, 10129 Torino, Italy

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Abstract

The theory of analytical functions is used to study the heat transfer from a uniformly heated cylinder with large length to diameter ratio in cross-flow, in the limit of small Péclet numbers. The energy conservation equation is solved in Fourier's space, and inverted by means of the residue theorem to obtain an analytical expression of the average Nusselt number in closed form. The result agrees with other theoretical solutions of the same problem existing in the literature.

1. Introduction

Heat transfer between a cylinder and a fluid stream in cross-flow is relevant in several practical applications. Examples are tube bundles in heat exchangers, but also hot-wire anemometers, submarine pipelines, electric and telecommunication cables, and many others. The heat transfer process is greatly affected by the quite complex hydrodynamic problem arising from the interaction between the cylinder and the flow, with the development of a boundary layer on the surface facing the flow and its subsequent separation in the rear part of the cylinder [1]. As it is well known, this leads to a non-uniform heat transfer coefficient [2], which is strongly influenced by the nature of boundary layer development.

From the standpoint of engineering calculations one is more interested in overall average conditions, so that the average heat transfer coefficient on the cylinder surface is required. The overall heat transfer coefficient is usually calculated by means of empirical correlations, which express the average Nusselt number as a function of the Reynolds and the Prandtl numbers of the fluid flow [3], and can be used in a given range of flow conditions. Unfortunately, uncertainties associated with the measurement of fluid velocities, the estimation of boundary effects at the cylinder ends, and the averaging of surface temperature, which varies both circumferentially and axially, render the experimental results accurate no better than 20–30%. As a consequence, calculations relying on empirical correlations are within the experimental uncertainty of the measurements on which they are based.

A case of special relevance in applications is that of uniformly heated cylinders with a large length to diameter ratio: examples are, among others, hot-wire anemometers and accidentally burning wires or cables. Following the pioneering work of King [4], this problem received much attention by researchers from the experimental, the theoretical and, more recently, the computational point of view.

Experimental studies generally show a large scatter of measured data, both at high values of the Reynolds number [4–7] and at smaller ones [8–10]. This is often a consequence of limitations in measurement accuracy, of the inappropriate treatment of the variation of fluid properties with temperature, of slip effects due to the small size of wires and, in the range of very small Reynolds numbers ($Re < 0.1$), it is also a consequence of the effects of natural convection. Thus, analytical and numerical approaches are

* Corresponding author.
E-mail address: v.bertola@ed.ac.uk (V. Bertola).

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Nomenclature

c_p	specific heat at constant pressure, J kg^{-1}K^{-1}
g	gravity acceleration, m s^{-2}
h	convection heat transfer coefficient, W m^{-2} K^{-1}
k	thermal conductivity, W m^{-1} K^{-1}
K_0	modified Bessel function of second kind
L	length, m
Nu	Nusselt's number ($Nu_{\mathrm{L}} = \frac{hL}{k}$)
P	pressure, Pa
Pe	Péclet's number ($Pe_{\mathrm{L}} = \frac{\rho u c_p L}{k}$)
Pr	Prandtl's number ($Pr = \frac{c_p \eta}{k}$)
R	cylinder radius, m
t	time, s
T'	temperature, K
T	dimensionless temperature
u	fluid stream velocity, m s^{-1}
w	dimensionless heat rate
W	heat rate per unit length, W m^{-1}
x	dimensionless spatial coordinate parallel to the flow
X	spatial coordinate parallel to the flow, m
y	dimensionless spatial coordinate perpendicular to the flow
Y	spatial coordinate perpendicular to the flow, m
z	complex variable
Z	spatial coordinate parallel to the wire, m
Greek symbols	
Γ	Euler's constant
$\delta(\bullet)$	delta function
η	viscosity, Pa s
λ	transformed dimensionless spatial variable in Fourier space
ρ	density, kg m^{-3}
θ	Fourier transform of the dimensionless temperature
ω	transformed dimensionless spatial variable in Fourier space
ζ	real variable
Subscripts	
∞	asymptotic conditions of the fluid stream
f	film
w	wall
R	referred to the cylinder radius

very useful, because they allow isolating the various effects and investigating them separately.

In particular, theoretical works addressed at first the solution of the flow field around the cylinder at small Reynolds numbers [11,12], which provided the starting point to solve the heat transfer problem [13–16]. These works showed that for small Péclet numbers, when diffusion dominates over advection, the details of the near-field become irrelevant and heat transfer depends on the characteristics of the far-field. Extensive and detailed reviews on the subject can be found in the recent literature [17,18].

Here, an approach based on the theory of analytical functions is proposed to obtain a mathematical solution in closed form for the heat transfer coefficient of a cylinder in cross-flow in the range $0 < Pe < 1$. The result is completely equivalent to other well-known theoretical solutions of the same problem that can be found in the open literature.

2. Analysis

Let's consider a linear heat source, such as a heated wire, placed in a gas stream uniformly flowing in the X-direction, perpendicular to the source, with constant velocity, u. The heat source is characterized by a linear heat emission density W (heat rate per unit length), and it is assumed to be aligned with the Z-axis, so that all parameters are Z-independent. Let's assume that all processes are steady and occur at constant pressure, and that the presence of the heat source does not perturb streamlines, which remain parallel to the X-axis. This assumption is equivalent to neglecting the near-field, which is a reasonable one for small Péclet numbers.

The continuity equation allows one to write:

$$\rho u = \rho_\infty u_\infty = \text{const} \tag{1}$$

where ρ_∞ and u_∞ are the asymptotic gas density and velocity. If the gas has constant thermophysical properties, the energy equation reduces to:

$$\rho_\infty c_p u_\infty \frac{\partial T'}{\partial X} = k\nabla^2 T' + W\delta(X)\delta(Y) \tag{2}$$

where c_p is the specific heat at constant pressure, k is the thermal conductivity coefficient, and δ is the delta function. The average heat transfer coefficient between the cylinder and the gas stream, h, is given by the relation:

$$W = 2\pi h R (T_{\mathrm{w}} - T_\infty) \tag{3}$$

where R is the radius of the wire, T_{w} is the temperature of the wire surface, and T_∞ is the asymptotic temperature of the gas.

The problem can be put into a dimensionless form by introducing the following quantities:

$$\begin{aligned} & x = \frac{X}{R}, \quad y = \frac{Y}{R} \\ & T = \frac{T' - T_\infty}{T_{\mathrm{w}} - T_\infty} \\ & w = \frac{W}{k(T_{\mathrm{w}} - T_\infty)}, \quad Nu_{\mathrm{R}} = \frac{hR}{k} \\ & Pe_{\mathrm{R}} = \frac{\rho_\infty u_\infty c_p R}{k} \\ & \delta(x) = R\delta(X), \quad \delta(y) = R\delta(Y) \end{aligned} \tag{4}$$

Then, Eqs. (2) and (3) can be re-written as

$$\frac{\partial T}{\partial x} = \frac{1}{Pe_R}[\nabla^2 T + w\delta(x)\delta(y)] \tag{5}$$

$$w = 2\pi Nu_R \tag{6}$$

Taking the Fourier transform of the energy equation one obtains

$$i\lambda\theta(\lambda,\omega) = -\frac{\lambda^2+\omega^2}{Pe_R}\theta(\lambda,\omega) + \frac{w}{2\pi Pe_R} \tag{7}$$

where λ and ω are complex variables, and

$$\theta(\lambda,\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\int_{-\infty}^{+\infty} T(x,y)\exp(-\mathrm{i}\lambda x - \mathrm{i}\omega y)\mathrm{d}y \tag{8}$$

Eq. (7) yields

$$\theta(\lambda,\omega) = \frac{w}{2\pi(\lambda^2 + \mathrm{i}\lambda Pe_R + \omega^2)} \tag{9}$$

The temperature distribution in the physical space of dimensionless coordinates x and y is obtained from Fourier inversion of Eq. (9)

$$\begin{aligned} T(x,y) &= \frac{w}{4\pi^2}\int_{-\infty}^{+\infty}\mathrm{d}\omega\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\lambda x + \mathrm{i}\omega y)}{\lambda^2+\mathrm{i}\lambda Pe_R+\omega^2}\mathrm{d}\lambda \\ &= \frac{w}{4\pi^2}\int_{-\infty}^{+\infty}\exp(\mathrm{i}\omega y)\mathrm{d}\omega\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\lambda x)}{\lambda^2+\mathrm{i}\lambda Pe_R+\omega^2}\mathrm{d}\lambda \end{aligned} \tag{10}$$

Eq. (10) can be integrated once over the λ-variable by means of the residue theorem [19]. In particular, since the integrand function $f(\lambda)$ satisfies Jordan's lemma, we have that for $x > 0$

$$\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\lambda x)}{\lambda^2+\mathrm{i}\lambda Pe_R+\omega^2}\mathrm{d}\lambda = 2\pi i\sum_k \mathrm{Res}\{f(z);\lambda_k\} \tag{11}$$

where λ_k are the poles of $f(\lambda)$ contained in the upper-half of the complex plane, that is, $\mathrm{Im}(\lambda_k) > 0$, and the residue is given by

$$\mathrm{Res}\{f(z);\lambda_k\} = (z-\lambda_k)f(z)|_{z=\lambda_k} \tag{12}$$

Analogously, for $x < 0$ we have that

$$\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\lambda x)}{\lambda^2+\mathrm{i}\lambda Pe_R+\omega^2}\mathrm{d}\lambda = -2\pi\mathrm{i}\sum_k \mathrm{Res}\{f(z);\lambda_k\} \tag{13}$$

where $\mathrm{Im}(\lambda_k) < 0$. The function $f(\lambda)$ has two first-order poles

$$\lambda_{1,2} = -\mathrm{i}\left(\frac{Pe_R}{2} \pm \sqrt{\frac{Pe_R^2}{4}+\omega^2}\right) \tag{14}$$

These poles are purely imaginary, and lay in the lower- and upper-half of the complex plane, respectively, so that Eqs. (10), (11) and (13) yield

$$T(x,y) = \begin{cases} \frac{w}{4\pi}\int_{-\infty}^{+\infty}\exp(\mathrm{i}\omega y)\dfrac{\exp\left[x\left(\frac{Pe_R}{2}-\sqrt{\frac{Pe_R^2}{4}+\omega^2}\right)\right]}{\sqrt{\frac{Pe_R^2}{4}+\omega^2}}\mathrm{d}\omega & x > 0 \\ \frac{w}{4\pi}\int_{-\infty}^{+\infty}\exp(\mathrm{i}\omega y)\dfrac{\exp\left[x\left(\frac{Pe_R}{2}+\sqrt{\frac{Pe_R^2}{4}+\omega^2}\right)\right]}{\sqrt{\frac{Pe_R^2}{4}+\omega^2}}\mathrm{d}\omega & x < 0 \end{cases} \tag{15}$$

For $x = 0$, the temperature can be expressed as

$$T(0,y) = \frac{w}{4\pi}\int_{-\infty}^{+\infty}\frac{\exp(\mathrm{i}\omega y)}{\sqrt{\frac{Pe_R^2}{4}+\omega^2}}\mathrm{d}\omega \tag{16}$$

Since on the wire surface we have that $T(0,\pm 1) = 1$, Eq. (16) can be written as

$$\frac{2\pi}{w} = \int_0^{+\infty}\frac{\cos(\omega)}{\sqrt{\frac{Pe_R^2}{4}+\omega^2}}\mathrm{d}\omega \tag{17}$$

Integrating the right hand side of Eq. (17) yields a modified Bessel function of second kind [20], so that

$$\frac{2\pi}{w} = K_0\left(\frac{Pe_R}{2}\right) \tag{18}$$

According to Eqs. (18) and (4), one finds the expression of Nu_R

$$Nu_R = \frac{1}{K_0\left(\frac{Pe_R}{2}\right)} \tag{19}$$

Referring Nusselt's and Péclet's dimensionless groups to the cylinder diameter, as it is standard practice in the literature, Eq. (19) can be re-written as

$$Nu = \frac{2}{K_0\left(\frac{Pe}{4}\right)} \tag{20}$$

Values of the function $K_0(\zeta)$ and of the corresponding Nusselt number are reported in Table 1.

Table 1
Values of the modified Bessel function of second kind $K_0(\zeta)$

ζ	$K_0(\zeta)$	$Nu = 2/K_0(\zeta)$
0.00	∞	0
0.05	3.1142	0.6422
0.10	2.4271	0.8240
0.15	2.0300	0.9852
0.20	1.7527	1.1411
0.25	1.5415	1.2974

3. Discussion

The validity of Eq. (20) is not generalized, but is obviously related to the assumptions made during its derivation. In particular, we have that:

1. The problem was treated in two dimensions, with no boundary effects considered at the two ends of the cylinder. Thus, the cylinder aspect ratio L/R cannot be too small ($L/R \gg 10$).
2. The thermophysical properties of the fluid are assumed to be uniform in space, whereas the fluid temperature is a function of the spatial coordinates. To account for the differences due to the temperature gradient, the fluid properties should be evaluated at the so-called mean film temperature $T_f = (T_w + T_\infty)/2$.
3. As mentioned above, assuming that streamlines are parallel to the X-axis, therefore neglecting their deformation in the vicinity of the cylinder surface, means that diffusion is the dominating mass transport mechanism in the flow around the cylinder: thus, the validity of the present approach is limited to small values of Péclet's number ($0 < Pe < 1$).

An assessment of Eq. (20) can be obtained from the comparison with some theoretical solutions of the heat transfer problem at small Reynolds numbers existing in the open literature. Cole and Roshko [13] applied the Oseen–Lamb solution for the flow [11,12] to the thermal energy equation (i.e., they linearized the inertia term of the flow equations at infinity), for small temperature differences (constant fluid properties). They found the first term of an expansion series in $[\ln(Pe)]^{-1}$ for the Nusselt number

$$Nu = \frac{2}{\ln\left(\frac{8}{Pe}\right) - \Gamma} \tag{21}$$

where $\Gamma \approx 0.5772$ is the Euler constant. Later, Nakai and Okazaki [16] matched two solutions for the temperature field, one corresponding to pure conduction in the vicinity of the cylinder and the other to a similarity solution for convection in the far field. They obtained the following expression:

$$Nu = \frac{2}{\frac{2}{3} + \ln\left(\frac{8}{3}\right) - \ln(Pe)} \tag{22}$$

In the proposed approach, the flow field in the vicinity of the cylinder is deliberately neglected, because its contribution vanishes in the limit $Pe \to 0$. This allows one to obtain a solution in closed analytical form (Eq. (20)), instead of truncated solutions. Fig. 1 shows that in spite of that approximation Eq. (20) is in very good agreement with both of these analytical solutions, and in practice cannot be distinguished from the Cole–Roshko solution for $Pe < 0.5$.

As expected, all solutions converge to a same value for $Pe \to 0$, because in this case any differences in modelling the flow field become irrelevant. When the Péclet number grows, the solutions shown in Fig. 1 diverge, as the flow field is taken into account to different degrees of approximation.

It is also of interest the comparison of Eq. (20) with some empirical correlations for cylinders in cross-flow. Here, we consider the well-known correlation proposed by Hilpert [5]:

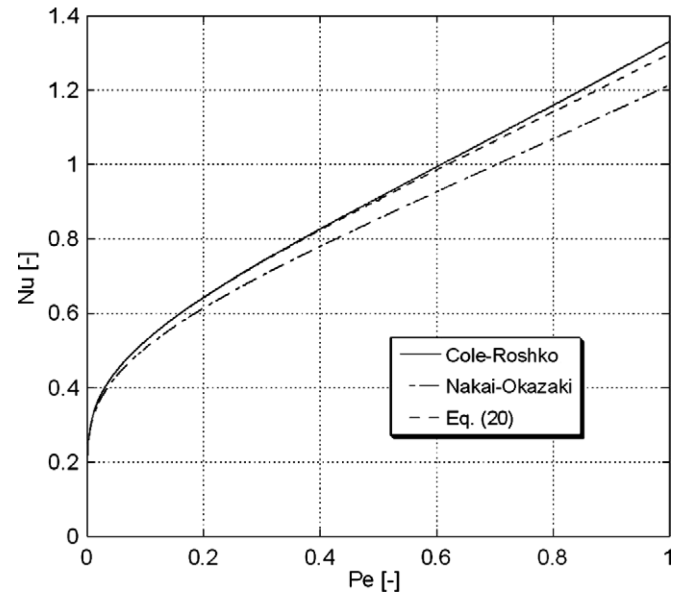

Fig. 1. Comparison of the proposed solution for the Nusselt number (Eq. (20)) with the solutions of Cole–Roshko (Eq. (21)) and Nakai–Okazaki (Eq. (22)).

$$Nu = CRe^m Pr^{1/3} = CPe^{1/3} Re^n \tag{23}$$

where the values of C and m (or C and n) depend on the Reynolds number, and the fluid properties are evaluated at the film temperature T_f. Fig. 2 shows that, in the range $0 < Pe < 1$, the proposed correlation exhibits a similar trend as long as the flow is laminar ($Re < 2000$).

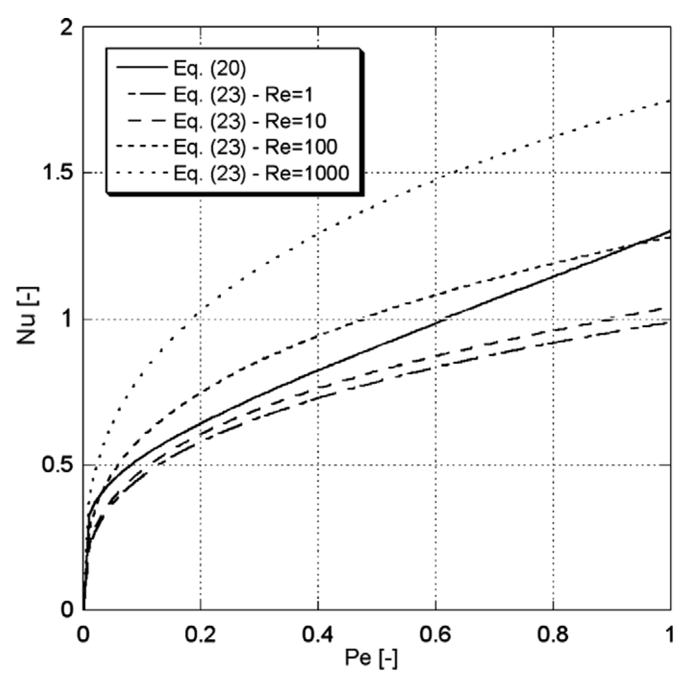

Fig. 2. Comparison between the proposed formula (Eq. (20)) and Hilpert's correlation (Eq. (23)), for different values of the Reynolds number.

4. Conclusions

The heat transfer problem between a uniformly heated cylinder with large length to diameter ratio and a fluid stream in cross-flow has been solved, in the limit of small Péclet numbers ($0 < Pe < 1$), using the theory of analytical functions. This approach allows one to obtain an exact solution of the problem in a closed analytical form, which returns the average Nusselt number as a monotonically growing function of Péclet's number.

The proposed solution is in very good agreement with other theoretical solutions of the same problem existing in the literature, which are based on the analysis of the flow field around the cylinder. Moreover, in the range of Péclet numbers considered, the present result exhibits the same trend as some well-known empirical correlations for the heat transfer coefficient of cylinders in cross-flow.

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